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# SOME IDENTITIES OF FUBINI POLYNOMIALS ARISING FROM DIFFERENTIAL EQUATIONS

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Abstract. Recently, Kim-Kim have studied degenerate Fubini polynomials, which are related to Stirling numbers of the second kind (see [6]).
In this paper, we find the differential equation arising from the generating function of Fubini polynomials and we derive some new interesting identities and properties from this differential equation.

## 1. Introduction

As is well known that the Fubini polynomials are defined by the generating function to be

$$\frac{1}{1-y(e^t-1)}=\sum_{n=0}^{\infty}F_n(y)\frac{t^n}{n!}. \tag{1.1}$$

Let $y=1$, Then we get

$$\frac{1}{2-e^t}=\sum_{n=0}^{\infty}F_n(1)\frac{t^n}{n!}. \tag{1.2}$$

When $y=1$, $F_n=F_n(1)$ are called the Fubini numbers.
The Fubini numbers are $F_0=1$, $F_1=1$, $F_2=3$, $F_3=13$, $F_4=75,\cdots$
Also, as is well known that the ordered bell numbers are defined by the generating function to be

$$\frac{1}{2-e^t}=\sum_{n=0}^{\infty}b_n\frac{t^n}{n!}. \tag{1.3}$$

From (1.2) and (1.3), we get

$$F_n(1)=b_n, \quad (n=0,1,2,\cdots). \tag{1.4}$$

From the left side of (1.1), we note that

$$\begin{aligned}\frac{1}{1-y(e^t-1)} &= \sum_{l=0}^{\infty} y^l (e^t-1)^l \\ &= \sum_{l=0}^{\infty} \frac{y^l}{l!} l! (e^t-1)^l \\ &= \sum_{l=0}^{\infty} \frac{y^l}{l!} \sum_{n=0}^{\infty} S_2(n,l) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \Big( \sum_{l=0}^{n} y^l l! S_2(n,l) \Big) \frac{t^n}{n!}.\end{aligned} \tag{1.5}$$

Therefore, by (1.1) and (1.5), we get

$$F_n(y) = \sum_{l=0}^{n} y^l l! S_2(n,l). \tag{1.6}$$

Now, we define the higher-order Fubini polynomials which are given by the generating function to be

$$\Big( \frac{1}{1-y(e^t-1)} \Big)^k = \sum_{n=0}^{\infty} F_n^{(k)}(y) \frac{t^n}{n!}, \quad (k \in \mathbb{N}). \tag{1.7}$$

Recently, several authors have studied Fubini numbers and polynomials (see[3,6,14-16]). In this paper, we study some nonlinear differential equations which are derived from the generating functions of Fubini polynomials. In addition, we give some new identities for the higher-order Fubini polynomials which are related to Fubini polynomials.

## 2. Some identities of Fubini polynomials arising from differential equation

Let

$$Y(t,y) = \frac{1}{1-y(e^t-1)}. \tag{2.1}$$

Then by taking the derivative with respect to $t$ of (2.1), we get

$$\begin{aligned}Y^{(1)} = \frac{\partial}{\partial t} Y(t,y) &= \frac{1}{\big(1-y(e^t-1)\big)^2} y e^t \\ &= \frac{1}{\big(1-y(e^t-1)\big)^2} \big((y(e^t-1)-1+1+y\big) \\ &= -Y + (1+y) Y^2.\end{aligned} \tag{2.2}$$

From (2.2), we have

$$\begin{aligned} Y^{(2)} &= -Y^{(1)} + 2(1+y)YY^{(1)} \\ &= Y^{(1)}\big(-1 + 2(1+y)Y\big) \\ &= \big(-Y + (1+y)Y^2\big)\big(-1 + 2(1+y)Y\big) \\ &= Y - 3(1+y)Y^2 + 2(1+y)^2Y^3. \end{aligned} \tag{2.3}$$

From (2.3), we note that

$$\begin{aligned} Y^{(3)} &= Y^{(1)} - 6(1+y)YY^{(1)} + 6(1+y)^2Y^2Y^{(1)} \\ &= Y^{(1)}\big(1 - 6(1+y)Y + 6(1+y)^2Y^2\big) \\ &= \big(-Y + (1+y)Y^2\big)\big(1 - 6(1+y)Y + 6(1+y)^2Y^2\big) \\ &= -Y + 7(1+y)Y^2 - 12(1+y)^2Y^3 + 6(1+y)^3Y^4. \end{aligned} \tag{2.4}$$

Continuing this process, we get

$$Y^{(N)} = \sum_{k=0}^{N}(-1)^{N-k}a_k(N)(1+y)^kY^{k+1}. \tag{2.5}$$

Let us take the derivative on the both sides of (2.5) with respect to $t$, Then we have

$$Y^{(N+1)} = \sum_{k=0}^{N}(-1)^{N-k}(k+1)a_k(N)(1+y)^kY^kY^{(1)}. \tag{2.6}$$

By (2.2) and (2.6), we get

$$\begin{aligned} Y^{(N+1)} &= \sum_{k=0}^{N}(-1)^{N-k}(k+1)a_k(N)(1+y)^kY^k\big(-Y + (1+y)Y^2\big) \\ &= \sum_{k=0}^{N}(-1)^{N-k+1}(k+1)a_k(N)(1+y)^kY^{k+1} + \sum_{k=0}^{N}(-1)^{N-k} \\ &\times (k+1)a_k(N)(1+y)^{k+1}Y^{k+2} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{N}(-1)^{N-k+1}(k+1)a_k(N)(1+y)^kY^{k+1} + \sum_{k=1}^{N+1}(-1)^{N-k+1}k \\
&\times a_{k-1}(N)(1+y)^kY^{k+1} \\
&= (-1)^{N+1}a_0(N)Y + (N+1)a_N(N)(1+y)^{N+1}Y^{N+2} \\
&+ \sum_{k=1}^{N}(-1)^{N-k+1}(k+1)a_k(N)(1+y)^kY^{k+1} + \sum_{k=1}^{N}(-1)^{N-k+1}k \\
&\times a_{k-1}(N)(1+y)^kY^{k+1} \\
&= (-1)^{N+1}a_0(N)Y + (N+1)a_N(N)(1+y)^NY^{N+1} \\
&+ \sum_{k=1}^{N}(-1)^{N-k+1}\big((k+1)a_k(N) + ka_k(N)\big)(1+y)^kY^{k+1}.
\end{aligned}
\tag{2.7}
$$

By replacing $N$ by $N+1$ in (2.5), we get

$$
\begin{aligned}
Y^{(N+1)} &= \sum_{k=0}^{N+1}(-1)^{N-k+1}a_k(N+1)(1+y)^kY^{k+1} \\
&= (-1)^{N+1}a_0(N+1)Y + a_{N+1}(N+1)(1+y)^{N+1}Y^{N+2} \\
&+ \sum_{k=1}^{N}(-1)^{N-k+1}a_k(N+1)(1+y)^kY^{k+1}.
\end{aligned}
\tag{2.8}
$$

Comparing the coefficients on the both sides of (2.7) and (2.8), we have

$$
a_0(N+1) = a_0(N), \quad a_{N+1}(N+1) = (N+1)a_N(N), \tag{2.9}
$$

and

$$
a_k(N+1) = (k+1)a_k(N) + ka_{k-1}(N), \tag{2.10}
$$

where $1 \le k \le N$.

From (2.2) and (2.5), we get

$$
\begin{aligned}
Y^{(1)} &= \sum_{k=0}^{1}(-1)^{1-k}a_k(1)(1+y)^kY^{k+1} \\
&= -a_0(1)Y + a_1(1)(1+y)Y^2 \\
&= -Y + (1+y)Y^2.
\end{aligned}
\tag{2.11}
$$

By (2.11), we get

$$
a_0(1) = 1, \quad a_1(1) = 1. \tag{2.12}
$$

Thus, by (2.9) and (2.12), we have

$$a_0(N+1) = a_0(N) = a_0(N-1) = \cdots = a_0(1) = 1, \tag{2.13}$$

and

$$\begin{aligned} a_{N+1}(N+1) &= (N+1)a_N(N) = (N+1)Na_{N-1}(N-1) = \cdots \\ &= (N+1)N\cdots 2a_1(1) = (N+1)N\cdots 2\cdot 1 = (N+1)!. \end{aligned} \tag{2.14}$$

From (2.10), we have

$$\begin{aligned} a_k(N+1) &= (k+1)a_k(N) + ka_{k-1}(N) \\ &= (k+1)\big((k+1)a_k(N-1) + ka_{k-1}(N-1)\big) + ka_{k-1}(N) \\ &= (k+1)^2 a_k(N-1) + (k+1)ka_{k-1}(N-1) + ka_{k-1}(N) \\ &= (k+1)^2\big((k+1)a_k(N-2) + ka_{k-1}(N-2)\big) \\ &+ (k+1)ka_{k-1}(N-1) + ka_{k-1}(N) \\ &= (k+1)^3 a_k(N-2) + (k+1)^2 ka_{k-1}(N-2) \\ &+ (k+1)ka_{k-1}(N-1) + ka_{k-1}(N) \\ &= \cdots \\ &= (k+1)^{N-k+1}a_k(k) + (k+1)^{N-k}ka_{k-1}(k) + \cdots \\ &+ ka_{k-1}(N). \end{aligned} \tag{2.15}$$

By (2.14) and (2.15), we get

$$\begin{aligned} a_k(N+1) &= (k+1)^{N-k+1}a_k(k) + (k+1)^{N-k}ka_{k-1}(k) + \cdots \\ &+ ka_{k-1}(N) \\ &= (k+1)^{N-k+1}ka_{k-1}(k-1) + (k+1)^{N-k}ka_{k-1}(k) \\ &+ \cdots + ka_{k-1}(N) \\ &= \sum_{i_1=0}^{N-k+1} (k+1)^{N-k+1-i_1} ka_{k-1}(k-1+i_1) \\ &= \sum_{i_1=0}^{N-k+1} (k+1)^{N-k+1-i_1} k \sum_{i_2=0}^{i_1} k^{i_1-i_2}(k-1) \end{aligned}$$

$$\begin{aligned}
&\times a_{k-2}(k-2+i_2)\\
&=\sum_{i_1=0}^{N-k+1}\sum_{i_2=0}^{i_1}(k+1)^{N-k+1-i_1}k^{i_1-i_2+1}(k-1)\\
&\times a_{k-2}(k-2+i_2)\\
&=\sum_{i_1=0}^{N-k+1}\sum_{i_2=0}^{i_1}(k+1)^{N-k+1-i_1}k^{i_1-i_2+1}(k-1)\\
&\times \sum_{i_3=0}^{i_2}(k-1)^{i_2-i_3}(k-2)a_{k-3}(k-3+i_3)\\
&=\sum_{i_1=0}^{N-k+1}\sum_{i_2=0}^{i_1}\sum_{i_3=0}^{i_2}(k+1)^{N-k+1-i_1}k^{i_1-i_2+1}(k-1)^{i_2-i_3+1}\\
&\times (k-2)a_{k-3}(k-3+i_3)\\
&=\cdots\\
&=\sum_{i_1=0}^{N-k+1}\sum_{i_2=0}^{i_1}\sum_{i_3=0}^{i_2}\cdots\sum_{i_k=0}^{i_{k-1}}(k+1)^{N-k+1-i_1}k^{i_1-i_2+1}\\
&\times (k-1)^{i_2-i_3+1}\cdots 2^{i_{k-1}-i_k+1}1\cdot a_0(i_k).
\end{aligned} \tag{2.16}$$

By (2.13) and (2.16), we get

$$\begin{aligned}
a_k(N+1)=&\sum_{i_1=0}^{N-k+1}\sum_{i_2=0}^{i_1}\cdots\sum_{i_k=0}^{i_{k-1}}(k+1)^{N-k+1-i_1}k^{i_1-i_2+1}\\
&\times\cdots\times 2^{i_{k-1}-i_k+1}.
\end{aligned} \tag{2.17}$$

**Theorem 2.1.** *Let $N \in \mathbb{N}\cup\{0\}$.*
*Then the following differential equations*

$$Y^{(N)}=\sum_{k=0}^{N}(-1)^{N-k}a_k(N)(1+y)^kY^{k+1},$$

*have a solution for $Y(t,y)=\frac{1}{1-y(e^t-1)}$, where*

$$\begin{aligned}
&a_0(N)=1,\quad a_N(N)=N!,\\
&a_k(N)=\sum_{i_1=0}^{N-k}\sum_{i_2=0}^{i_1}\cdots\sum_{i_k=0}^{i_{k-1}}(k+1)^{N-k-i_1}k^{i_1-i_2+1}\times\cdots\times 2^{i_{k-1}-i_k+1}
\end{aligned}$$

*for $1\le k\le N-1$.*

By (1.1), we have

$$\begin{aligned} Y^{(N)} &= \left(\frac{\partial}{\partial t}\right)^N Y(t,y) \\ &= \left(\frac{\partial}{\partial t}\right)^N \Big(\sum_{n=0}^{\infty} F_n(y)\frac{t^n}{n!}\Big) \\ &= \sum_{n=0}^{\infty} F_{n+N}(y)\frac{t^n}{n!}. \end{aligned} \tag{2.18}$$

Thus by (1.7), (2.5) and (2.18), we get

$$\begin{aligned} \sum_{n=0}^{\infty} F_{n+N}(y)\frac{t^n}{n!} &= \sum_{k=0}^{N}(-1)^{N-k}a_k(N)(1+y)^k \sum_{n=0}^{\infty} F_n^{(k+1)}(y)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{N}(-1)^{N-k}a_k(N)(1+y)^k F_n^{(k+1)}(y)\Big)\frac{t^n}{n!}. \end{aligned} \tag{2.19}$$

**Theorem 2.2.** *For $n, N \in \mathbb{N} \cup \{0\}$, we have*

$$F_{n+N}(y) = \sum_{k=0}^{N}(-1)^{N-k}a_k(N)(1+y)^k F_n^{(k+1)}(y).$$

Now, we consider the inversion formula of Theorem1
From (2.2), we get

$$(1+y)Y^2 = Y + Y^{(1)}. \tag{2.20}$$

take the derivative on the both sides of (2.20), we get

$$2(1+y)YY^1 = Y^{(1)} + Y^{(2)}. \tag{2.21}$$

By (2.2) and (2.21), we get

$$2(1+y)Y\big(-Y+(1+y)Y^2\big) = Y^{(1)} + Y^{(2)}. \tag{2.22}$$

From (2.22), we get

$$\begin{aligned} 2(1+y)^2Y^3 &= 2(1+y)Y^2 + Y^{(1)} + Y^{(2)} \\ &= 2Y + 3Y^{(1)} + Y^{(2)}. \end{aligned} \tag{2.23}$$

From (2.23), we have

$$3!(1+y)^3Y^2Y^{(1)} = 2Y^{(1)} + 3Y^{(2)} + Y^{(3)}. \tag{2.24}$$

From (2.2) and (2.24), we get

$$3!(1+y)^3Y^4 = 3!(1+y)^2Y^3 + 2Y^{(1)} + 3Y^{(2)} + Y^{(3)}. \tag{2.25}$$

From (2.23) and (2.25), we get

$$\begin{aligned}3!(1+y)^3Y^4 &= 3(2Y+3Y^{(1)}+Y^{(2)})+2Y^{(1)}+3Y^{(2)}+Y^{(3)}\\ &= 6Y+11Y^{(1)}+6Y^{(2)}+Y^{(3)}.\end{aligned} \tag{2.26}$$

Continuing this process, we get

$$N!(1+y)^NY^{N+1}=\sum_{k=0}^{N}b_k(N)Y^{(k)}. \tag{2.27}$$

Let us take the derivative on the both sides of (2.27) with respect to $t$, Then we have

$$(N+1)!(1+y)^NY^NY^{(1)}=\sum_{k=0}^{N}b_k(N)Y^{(k+1)}. \tag{2.28}$$

Then by (2.2) and (2.28), we have

$$(N+1)!(1+y)^{N+1}Y^{N+2}=(N+1)!(1+y)^NY^{N+1}+\sum_{k=0}^{N}b_k(N)Y^{(k+1)}. \tag{2.29}$$

From (2.27) and (2.29), we get

$$\begin{aligned}(N+1)!(1+y)^{N+1}Y^{N+2} &= \sum_{k=0}^{N}(N+1)b_k(N)Y^{(k)}+\sum_{k=0}^{N}b_k(N)Y^{(k+1)}\\ &= \sum_{k=0}^{N}(N+1)b_k(N)Y^{(k)}+\sum_{k=1}^{N+1}b_{k-1}(N)Y^{(k)}\\ &= (N+1)b_0(N)Y+b_N(N)Y^{(N+1)}\\ &+\sum_{k=1}^{N}\big((N+1)b_k(N)+b_{k-1}(N)\big)Y^{(k)}.\end{aligned} \tag{2.30}$$

By replacing $N$ by $N+1$ in (2.27), we get

$$\begin{aligned}(N+1)!(1+y)^{N+1}Y^{N+2} &= \sum_{k=0}^{N+1}b_k(N+1)Y^{(k)}\\ &= b_0(N+1)Y+b_{N+1}(N+1)Y^{(N+1)}\\ &+\sum_{k=1}^{N}b_k(N+1)Y^{(k)}.\end{aligned} \tag{2.31}$$

Comparing the coefficients on the both sides of (2.30) and (2.31), we get

$$b_0(N+1)=(N+1)b_0(N),\quad b_{N+1}(N+1)=b_N(N). \tag{2.32}$$

and

$$b_k(N+1) = (N+1)b_k(N) + b_{k-1}(N), \tag{2.33}$$

for $1 \leq k \leq N$.

From (2.2) and (2.27), we get

$$\begin{aligned} (1+y)Y^2 &= \sum_{k=0}^{1} b_k(1)Y^{(k)} \\ &= b_0(1)Y + b_1(1)Y^{(1)} \\ &= Y + Y^{(1)}. \end{aligned} \tag{2.34}$$

From (2.34), we get

$$b_0(1) = 1, \quad b_1(1) = 1. \tag{2.35}$$

By (2.32) and (2.34), we get

$$\begin{aligned} b_0(N+1) &= (N+1)b_0(N) = (N+1)Nb_0(N-1) = \cdots \\ &= (N+1)N\cdots 2b_0(1) = (N+1)N\cdots 2\cdot 1 = (N+1)!, \end{aligned} \tag{2.36}$$

and

$$b_{N+1}(N+1) = b_N(N) = b_{N-1}(N-1) = \cdots = b_1(1) = 1. \tag{2.37}$$

By (2.33), we have

$$\begin{aligned} b_k(N+1) &= (N+1)b_k(N) + b_{k-1}(N) \\ &= (N+1)\big(Nb_k(N-1) + b_{k-1}(N-1)\big) + b_{k-1}(N) \\ &= (N+1)_2 b_k(N-1) + (N+1)_1 b_{k-1}(N-1) + b_{k-1}(N) \\ &= \cdots \\ &= (N+1)_{N-k+1} b_k(k) + (N+1)_{N-k} b_{k-1}(k) \\ &+ \cdots + b_{k-1}(N) \end{aligned} \tag{2.38}$$

From (2.32) and (2.38), we get

$$\begin{aligned} b_k(N+1) &= (N+1)_{N-k+1}b_k(k) + (N+1)_{N-k}b_{k-1}(k) \\ &+ \cdots + b_{k-1}(N) \\ &= (N+1)_{N-k+1}b_{k-1}(k-1) + (N+1)_{N-k}b_{k-1}(k) \\ &+ \cdots + b_{k-1}(N) \\ &= \sum_{i_1=0}^{N-k+1} (N+1)_{N-k+1-i_1} b_{k-1}(k-1+i_1) \\ &= \sum_{i_1=0}^{N-k+1} (N+1)_{N-k+1-i_1} \sum_{i_2=0}^{i_1} (k-1+i_1)_{i_1-i_2} b_{k-2}(k-2+i_2) \\ &= \sum_{i_1=0}^{N-k+1} \sum_{i_2=0}^{i_1} (N+1)_{N-k+1-i_1}(k-1+i_1)_{i_1-i_2} b_{k-2}(k-2+i_2) \\ &= \sum_{i_1=0}^{N-k+1} \sum_{i_2=0}^{i_1} (N+1)_{N-k+1-i_1}(k-1+i_1)_{i_1-i_2} \sum_{i_3=0}^{i_2} (k-2+i_2)_{i_2-i_3} \\ &\times b_{k-3}(k-3+i_3) \\ &= \sum_{i_1=0}^{N-k+1} \sum_{i_2=0}^{i_1} \sum_{i_3=0}^{i_2} (N+1)_{N-k+1-i_1}(k-1+i_1)_{i_1-i_2}(k-2+i_2)_{i_2-i_3} \\ &\times b_{k-3}(k-3+i_3) \\ &= \cdots \\ &= \sum_{i_1=0}^{N-k+1} \sum_{i_2=0}^{i_1} \sum_{i_3=0}^{i_2} \cdots \sum_{i_k=0}^{i_{k-1}} (N+1)_{N-k+1-i_1}(k-1+i_1)_{i_1-i_2} \\ &\times (k-2+i_2)_{i_2-i_3} \cdots (1+i_{k-1})_{i_{k-1}-i_k} b_0(i_k). \end{aligned} \tag{2.39}$$

By (2.36) and (2.39), we get

$$\begin{aligned} b_k(N+1) = &= \sum_{i_1=0}^{N-k+1} \sum_{i_2=0}^{i_1} \sum_{i_3=0}^{i_2} \cdots \sum_{i_k=0}^{i_{k-1}} (N+1)_{N-k+1-i_1}(k-1+i_1)_{i_1-i_2} \\ &\times (k-2+i_2)_{i_2-i_3} \cdots (1+i_{k-1})_{i_{k-1}-i_k} i_k!. \end{aligned} \tag{2.40}$$

**Theorem 2.3.** *For $N \in \mathbb{N} \cup \{0\}$, Then the following differential equations*

$$N!(1+y)^N Y^{N+1} = \sum_{k=0}^{N} b_k(N) Y^{(k)}.$$

*have a solution* $Y(t,y)=\frac{1}{1-y(e^t-1)}$, *where*

$$b_0(N)=N!,\quad b_N(N)=1.$$

*and*

$$\begin{aligned}b_k(N)==&\sum_{i_1=0}^{N-k}\sum_{i_2=0}^{i_1}\sum_{i_3=0}^{i_2}\cdots\sum_{i_k=0}^{i_{k-1}}(N)_{N-k-i_1}(k-1+i_1)_{i_1-i_2}\\ &\times(k-2+i_2)_{i_2-i_3}\cdots(1+i_{k-1})_{i_{k-1}-i_k}i_k!.\end{aligned}$$

By (1.7), (2.18) and (2.27), we get

$$\begin{aligned}N!(1+y)^N\sum_{n=0}^{\infty}F_n^{(N+1)}(y)\frac{t^n}{n!}&=\sum_{k=0}^{N}a_k(N)\sum_{n=0}^{\infty}F_{n+k}(y)\frac{t^n}{n!}\\ &=\sum_{n=0}^{\infty}\Big(\sum_{k=0}^{N}b_k(N)F_{n+k}(y)\Big)\frac{t^n}{n!}.\end{aligned}\tag{2.41}$$

Thus, by (2.41), we get

**Theorem 2.4.** *Let* $n,N\in\mathbb{N}\cup\{0\}$, *we get*

$$F_n^{(N+1)}(y)=\sum_{k=0}^{N}b_k(N)F_{n+k}(y).$$

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